MATH 512 NOTES ON FORCING

The ground model is the universe V. In V, $(\mathbb{P}, \leq)$ or just $\mathbb{P}$ will denote a partial ordering (*poset*) with a maximal element $1_{\mathbb{P}}$. Elements in $\mathbb{P}$ are called *conditions*. For two conditions p, q in $\mathbb{P}$, we say that:

- p is *stronger* than q if $p \leq q$,
- p and q are *compatible* if there is some r such that $r \leq p$ and $r \leq q$; r is called a *common extension*. Otherwise p, q are *incompatible*, and we write $p \perp q$.

A subset $D \subset \mathbb{P}$ is called *dense* if for every $p \in \mathbb{P}$, there is $q \leq p$ with $q \in D$. A dense set D is *dense open* if it is closed downwards. D is *dense below* p if for every $p' \leq p$, there is $q \leq p'$ with $q \in D$. A subset $A \subset \mathbb{P}$ is an *antichain* if every two distinct $p, q \in A$ are incompatible. An antichain A is a *maximal antichain* if every condition p is compatible with some element in A.

Definition 1. *Suppose that $\mathbb{P}$ is partial ordering. $G \subset \mathbb{P}$ is a* **generic filter** *for $\mathbb{P}$ over the ground model V if:*

1. *G is a filter i.e. $1_{\mathbb{P}} \in G$, G is closed upwards, and any two conditions p, q in G have a common extension r also in G.*
2. *(Genericity) For every dense $D \subset \mathbb{P}$ with $D \in V$, $G \cap D \neq \emptyset$.*

Assuming G is a filter, the second condition is equivalent to each of the following:

- For every open dense $D \subset \mathbb{P}$ with $D \in V$, $G \cap D \neq \emptyset$;
- For every maximal antichain $A \subset \mathbb{P}$ with $A \in V$, $G \cap A \neq \emptyset$.

We will say that $\mathbb{P}$ is the trivial poset if $\mathbb{P} = \{1\}$. Otherwise we assume that for all $p \in \mathbb{P}$, there are incompatible $q, r \leq p$. That guarantees that generic filters are not in the ground model. (Otherwise $\{p \mid p \notin G\}$ is a dense set in the ground model).

Names and generic extensions:

A $\mathbb{P}$-**name** τ in V is a relation, whose elements are of the form $\langle \sigma, p \rangle$, where σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. If G is a generic filter and τ is a $\mathbb{P}$-name, set $\tau_G = \{\sigma_G \mid (\exists p \in G)(\langle \sigma, p \rangle \in \tau)\}$. Then

$$V[G] := \{\tau_G \mid \tau \text{ is a } \mathbb{P} \text{ name}\}.$$

$V[G]$ is called a *generic extension*. It satisfies the following:

1. $V[G] \models ZFC$.
2. $V \subset V[G]$.
3. $G \in V[G] \setminus V$.
4. Actually $V[G]$ is the smallest model of ZFC containing V and G.

Some examples and notation:

(1) For $a \in V$, the canonical $\mathbb{P}$-name corresponding to it is $\check{a} := \{\langle \check{b}, 1_{\mathbb{P}}\rangle \mid b \in a\}$. Note that for any generic G, $\check{a}_G = a$. In a slight abuse of notation we often use simply a in place of $\check{a}$ when a is in V.
(2) For an element $a \in V[G]$, we will often use $\dot{a}$ to denote a name for a, such that $\dot{a}_G = a$.
(3) The canonical name for a generic filter of $\mathbb{P}$ is $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$. Note that the name is independent of G and for any generic filter G, $\dot{G}_G = G$.

The forcing relation

For a formula $\phi(v_1, ..., v_n)$, $\mathbb{P}$-names $\tau_1, ..., \tau_n$, and a condition p, we define a forcing relation,

$$p \Vdash \phi(\tau_1, ..., \tau_n)$$

in the ground model V, so that:

(1) $p \Vdash$ "$\sigma \in \tau$" iff $\{q \mid (\exists r)(q \leq r, \langle \sigma, r\rangle \in \tau)\}$ is dense below p.
(2) $p \Vdash$ "$\tau_1 = \tau_2$" iff for all names σ and for all $q \leq p$ we have

$$q \Vdash \text{“}\sigma \in \tau_1\text{”} \leftrightarrow q \Vdash \text{“}\sigma \in \tau_2\text{”}.$$

(3) $p \Vdash \neg\phi$ iff for all $q \leq p$, $q \nVdash \phi$.
(4) $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$.
(5) $p \Vdash (\exists x)\phi(x)$ iff $\{q \mid (\exists \dot{a})(q \Vdash \phi(\dot{a}))\}$ is dense below p.

We say that p *decides* ϕ if $p \Vdash \phi$ or $p \Vdash \neg\phi$. In that case we write $p \parallel \phi$. We also have the following:

(1) If $q \leq p$ and $p \Vdash \phi$, then $q \Vdash \phi$;
(2) For any ϕ, $\{p \in \mathbb{P} \mid p \parallel \phi\}$ is dense.

Examples:

- If ϕ holds is V, then it is forced by all conditions, i.e. $1_{\mathbb{P}} \Vdash \phi$.
- Let $\dot{G}$ be the canonical name for a generic filter defined above. Then for all conditions p, $p \Vdash p \in \dot{G}$.
- Let $\alpha < \beta$ be ordinals, $\tau = \{\langle \alpha, p\rangle\}$ and $p \perp q$. Then $p \Vdash \alpha \in \tau$ and $q \Vdash \tau = \emptyset$.
- Let $A = \{p_i \mid i < \mu\}$ be a maximal antichain and $\langle \alpha_i \mid i < \mu\rangle$ be an increasing sequence of ordinals unbounded in μ. Let $\tau := \{\langle \alpha_i, p_i\rangle \mid i < \mu\}$. Then $1_{\mathbb{P}} \Vdash$ "$\tau \subset \mu, |\tau| = 1$", but $1_{\mathbb{P}}$ does not decide what is the single element of τ.

Theorem 2. *(The Forcing Theorem)*

(1) *If G is $\mathbb{P}$-generic over V, $V[G] \models \phi(\tau^1_G, ..., \tau^n_G)$ iff there is $p \in G$, such that $p \Vdash \phi(\tau^1, ..., \tau^n)$*
(2) *$p \Vdash \phi(\tau^1, ..., \tau^n)$ iff for every generic filter G with $p \in G$, $V[G] \models \phi(\tau^1_G, ..., \tau^n_G)$.*